Elegant physics problems

Tien Nguyen Thanh

July 2026 (updated: 21st July 2026)

Abstract

There is a certain elegance to theoretical problems if they can be solved entirely in your head, without pen and paper. Naturally, this forces the solutions to be less calculation-prone and more geometric. In this writing, I present three such problems admitting elegant and unexpected solutions which one could come up with by pure thought. I learned these problems from Ilja Göthel, a prolific physics educator and coach for physics competitions, back in my student days.

1 The problems

First, we present the three problems. As is often the case, we assume various idealised conditions which are typical for physics problems:

- Any kind of friction or air resistance is to be neglected.
- All objects or masses are point masses i. e., infinitesimally small.
- All objects exist in infinite space and time.
- The earth has uniform density.
- The gravitational field of earth is uniform, i. e., earth is infinite and flat.

An exception will be the last problem, where we assume the earth to be a perfect sphere.

Problem 1. *A hunter is chasing after a rabbit. The rabbit starts at a point on the plane and always runs east with unit velocity. The hunter starts north of the rabbit at distance r, and at any point in time will run towards the rabbit with unit velocity. After infinite time, which distance from the rabbit will the hunter achieve?*

Problem 2. *Consider a bomb in form of a infinitesimal semi-sphere on the ground. Upon detonation, each splinter has the same velocity and initially moves radially away from the bomb. How big is the circle on the ground in which exactly half of all splinters will land? It suffices to find the ratio of the circle's radius to the maximum range of the splinters.*

Problem 3 (gravity train)**.** *Suppose that we dug a tunnel straight through earth until we reach the other side of earth's surface (the tunnel does not need to pass through the centre of the earth). If we drop a mass m inside that tunnel, the mass will exhibit oscillation. What can you say about the period of this oscillation?*

2 Solution to problem 1

The trajectory traced by the rabbit is quite complicated. Since there are no forces at play, the classic strategy is to choose an appropriate inertial frame of reference. In our case, we want to choose the reference frame of the rabbit. From its point of view, only the hunter moves with two unit velocity vectors: one always pointing towards the rabbit, and one always pointing westwards. Note that by changing the frame of reference, the trajectory of the hunter changes, but the distance between hunter and rabbit at any point in time is the same (distances are always relative).

We now want to determine the trajectory of the hunter. The sum of the two unit velocity vectors describes the tangent of the trajectory at the current position of the hunter. Since the unit vectors span a rhombus, the tangent bisects the angle spanned by the vectors.

Now comes the ingenious geometric argument: extending the vectors to straight lines, the normal (perpendicular) of the tangent is an angle bisector of the adjacent angle. In other words, the tangent is a mirror reflecting a ray of light coming from the right towards the rabbit! From optics, we know that a mirror reflecting parallel light towards a focal point must be parabolic. Hence, the trajectory of the hunter is a parabola as depicted in figure 1.

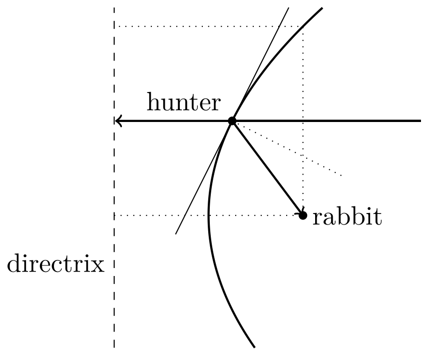

Figure 1: Trajectory of the hunter in the reference frame of the rabbit.

Geometrically, a parabola is the locus of all points equidistant from a straight line (directrix) and a point (focus). Initially, the hunter is r away from the rabbit, so the vertical straight line has distance r from the rabbit. After infinite time, the hunter is at the point of symmetry of the parabola, so precisely midway between directrix and rabbit. Thus:

Answer 1. *The distance between hunter and rabbit after infinite time is* $r/2$.

There is another relatively easy solution with calculation due to [1]. In the reference frame of the rabbit, let $z(t)$ and $x(t)$ be the distance and horizontal separation between hunter and rabbit, resp. After infinite time, we have $z = x$. To be clear, let $v = 1$ be the velocity of hunter and rabbit. For a small time step dt, we then have $dx(t) = (v - v\cos\theta)dt$ and $dz(t) = (v\cos\theta - v)dt$ where θ is the angle between the x-axis and the hunter. Hence, $dx(t) + dz(t) = 0$ for small dt, so $x(t) + z(t)$ must be constant for all t. Initially at $t = 0$, this quantity is r, so as $t \to \infty$, we have $x + z = 2x = r$.

3 Solution to problem 2

The initial urge is to only consider a radial slice through the bomb and calculate everything in two dimensions. The issue is that in such a picture, the splinters near the ground are much 'denser' than the splinters near the top as there is much more circumference of the bomb in the third dimension. In the extreme cases, the splinter in the slice at the ground represents a whole circle of splinters, while the splinter at the very top in the slice only represents a single splinter.

Maybe unexpectedly, a classical geometric result due to Archimedes comes to the rescue. He proved that the surface area of a sphere equals the lateral surface area of a cylinder, i. e., minus the top and base circles, of the same radius. Even more, this holds even infinitesimally in the following sense. Consider a sphere inscribed in a cylinder, and regard the cylinder as a cake. For a small slice of cake, and of that slice a small level section of that slice, see figure 2, the surface patches of the cylinder and the sphere have the same area. Intuitively, projecting a surface patch of the sphere onto the cylinder, the width scales by a factor r, but the height scales by a factor $1/r$, so both changes perfectly cancel. See the excellent video [2] for a more detailed account. Hence, it is justified to only consider a radial slice of the situation where after projecting the bomb horizontally onto the axis of rotation, the splinters are uniformly distributed on the line.

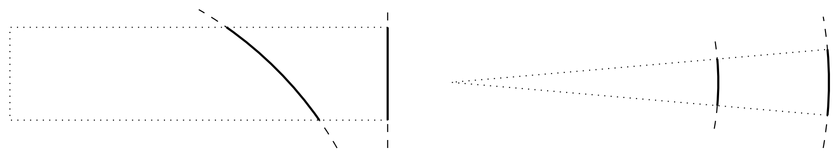

Figure 2: Level section of a slice of cake: left profile view, right bird's eye view.

Next, we should determine when two splinters on the circle land on the same spot. Intuitively, a very steep and a very shallow splinter can land on the same spot, while the splinter which reaches furthest is the one with launch angle 45°. For a splinter with velocity vector v, its range is determined by the horizontal component v_x and by its time airborne. The time of flight is proportional to the vertical component v_y (the time from launch up to reaching the apex is the time gravitation needs to stop the vertical velocity, and twice that is the time of flight). Hence, the range is (up to a constant) the area $v_x v_y$ of the rectangle spanned by v. Reflecting the rectangle at the line with inclination 45°, the roles of v_x and v_y swap, so splinters symmetric w. r. t. 45° have the same range. (There cannot be three splinters of the same range since the problem is a quadratic equation. Or viewed differently, a hyperbola can intersect a circle in at most two points.)

Assuming w. l. o. g. unit velocity, we are therefore interested in the unit vector v for which the reflection v' at 45° satisfies $v'_y - v_y = \frac{1}{2}$ since this range will be precisely half of all splinters landing outside of our desired circle. In other words, we are searching for a rectangle with unit diagonal v whose side lengths differ by $v_x - v_y = \frac{1}{2}$ and we only care about the area $A = v_x v_y$. Consider figure 3 and note that the doubly hatched square is counted twice. By the Pythagorean theorem, the whole square including the doubly hatched square is $v_x^2 + v_y^2 = v^2 = 1$. Hence, we have $A = \frac{3}{8}$. In relation to the square spanned by the unit vector at 45° of area $\frac{1}{2}$, we have:

Answer 2. *The radius of the circle with exactly half the splinters is* $\frac{1}{2}/\frac{3}{8} = \frac{3}{4}$ *of the maximum range of the splinters.*

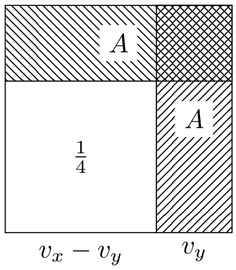

Figure 3: Visual representation of $(v_x - v_y)^2 = v_x^2 + v_y^2 - 2A$.

4 Solution to problem 3

We should first convince ourselves that this oscillation is harmonic. The crucial fact is that the gravitational force satisfies an inverse-square law: the force exerted by a point mass M of distance r is proportional to M/r^2. For this, we first prove Newton's shell theorem from astronomy. For a hollow sphere mass, the gravitational force exerted on

1. a point mass inside the sphere is none,
2. a point mass outside the sphere is the same as if the sphere were concentrated at the centre.

The first point is maybe unexpected. For a point inside a hollow sphere, consider a small double cone as in figure 4. Note that both cones are similar since for any slice through the central axis of the double cone, all the angles are the same by the inscribed angle theorem. If the ratio of the side lengths of the cones is λ, then the gravitational acceleration scales by $1/\lambda^2$, but the mass that is illuminated scales by λ^2. Hence, the gravitational acceleration inside the double cone perfectly cancels.

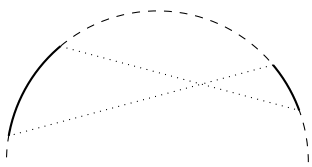

Figure 4: Newton's shell theorem inside a sphere.

The second one seems intuitively true, but is much harder to show. Consider a point outside of the hollow sphere M with distance r from the origin. By spherical symmetry, the gravitational acceleration experienced by any point on a sphere S of radius r is the same, so it suffices to average the gravitational flux through S over the area of S. The flux says how quickly a mass would accelerate perpendicular to S, weighted by area. Gauss's law for gravity states that the flux through S only depends on the total mass of M inside S, no matter the shape of M.

If Gauss's law is to be avoided, which relies on the divergence theorem, we reproduce a geometric argument due to [3]. We calculate the flux caused by a small part dM of M. Let $d\Omega$ be a small symmetric square patch tangent to the unit sphere around dM, and consider the pyramid over $d\Omega$ with apex dM in figure 5. By scaling the pyramid until the centre of $d\Omega$ lies on S, the area of of $d\Omega$ scales up by the square of this distance, but the gravitational acceleration scales down by the same amount. Next, take the radial component of the gravitational acceleration and take the 'intersection' of the pyramid with S. Then the acceleration scales down by value, but the surface area of the intersection dS scales up by the same value because of similarity of right triangles. Hence, the flux through $d\Omega$ is the same as the one through dS. Summing over all $d\Omega$, we see that the total flux through S is the same as the flux through the unit sphere around dM, and this does not depend on the position of dM. Moving all the dM to the centre, we obtain Gauss's law.

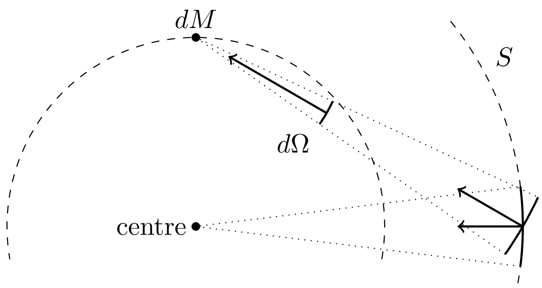

Figure 5: Proof of Gauss's law for gravity and for spherical surfaces.

Going back to our problem, we drop the mass m inside the tunnel. By the shell theorem, the gravitational force that m experiences is proportional to its distance r to earth's centre since the volume and hence the mass of earth is proportional to r^3. Projecting both the distance vector and force vector onto the path of the tunnel, we see that the oscillation satisfies Hooke's law: the restoring force on m towards the equilibrium is proportional to its displacement.

This already suggest transitioning to phase space: instead of m oscillating in the tunnel, we should let m orbit 'inside' earth where the orbit's diameter is the same as the length of the tunnel. We may recover the oscillation by projecting the motion onto the tunnel like in figure 6, so their periods agree.

By Kepler's third law of planetary motion, the square of the orbital period is proportional to the cube of the orbital radius r. It also is inverse proportional to the mass around which m orbits (the centripetal force, which is the gravitational force, is given by $mr\omega^2$ where ω is the angular velocity). But the mass is proportional to r^3 as we have established. Thus:

Answer 3. *The period is always the same, irrespective of the length of the tunnel.*

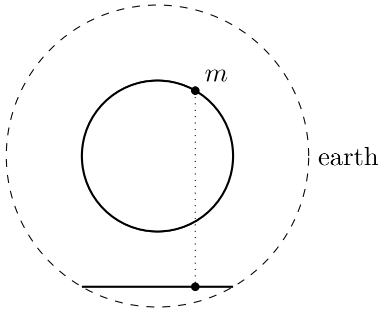

Figure 6: Projecting the orbital motion to the oscillation.

Using actual equations like

$$T^2 = \frac{4\pi^2 r^3}{GM(r)} = \frac{3\pi}{G\rho}$$

where $M(r)$ is the mass of the earth, ρ is its density and G is the gravitational constant, we obtain that $T \approx 2530.3\,\mathrm{s} \approx 42.2\,\mathrm{min}$.

References

[1] Kagan, Mikahil. *Non-Cartesian thinking.* arXiv (2011). URL: https://arxiv.org/abs/1108.2006.

[2] Sanderson, Grant (3Blue1Brown). *But why is a sphere's surface area four times its shadow?* YouTube (2nd December 2018). URL: https://www.youtube.com/watch?v=GNcFjFmqEc8.

[3] Schmid, Christoph. *Newton's superb theorem: An elementary geometric proof.* arXiv (2012). URL: https://arxiv.org/abs/1201.6534.