Turtles all the way down

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In an effort of relearning category theory, I stumbled across a problem, annoying at best and devastating at worst, that many do not take seriously enough when learning the subject. Its roots trace back to the foundations, throwing mathematics into great turmoil a century ago. I am talking about *size issues* which have far-reaching consequences in category theory and, for that reason, deserve greater attention. To wet your appetite: the classical Yoneda lemma only holds for *small* categories, i. e., whose objects form a set! By introducing so-called *Grothendieck universes*, one can almost remove all size restrictions, but only *almost*.

The text is directed at readers who are fairly well acquainted with the notions of categories, functors, natural transformations and limits. Keep in mind that I am not a foundationalist, so take the section on universes with a grain of salt.

Universes

Informally, a *universe* is the stage on which mathematical objects perform in a play called a "theory". Since most of mathematics is based on *Zermelo-Fraenkel set theory* with the *Axiom of choice* (ZFC), such a universe would be a "collection of all sets". Russell's paradox, however, tells us that:

> The "set of all sets" does not exist!

Otherwise, we could form the subset $S = \{T \mid T \notin T\}$ of all sets not containing themselves, which famously cannot exist (is S contained in itself or not?). There are essentially two approaches to resolve this issue.

Solution 1: The "set of all sets" is not a set

The first approach is the most widely adopted solution in mathematics. The idea is to have a second type of objects besides sets called *classes*, which are "larger" than sets. There are various axiomatisations of this idea, but the most popular is *von Neumann-Bernays-Gödel set theory* (NBG). My presentation of NBG here is based on the excellent Wikipedia article [8] which follows Gödel's theory from 1940. Recall that ZFC has sets as objects and is membership based, i. e., one can say $S \in T$ for two sets S and T.

Axiom (von Neuman-Bernays-Gödel set theory, NBG)

NBG is also membership based, but uses classes instead of sets. By definition, a class is a *set* if it is an element of another class, and is otherwise a *proper class*. Furthermore, NBG uses the following axioms:

- Equality of classes is defined identically to equality of sets in ZFC.
- The Axiom schema of restricted comprehension in ZFC is replaced by the *Class comprehension schema*: if ϕ is a statement on sets, then one can form the class of all sets satisfying ϕ. In ZFC, we are only allowed to form the subset $\{T \in S \mid \phi(T)\}$ of a given set S. It is important that ϕ quantifies over sets, not classes.
- The Axiom of choice is replaced by the stronger *Axiom of global choice*: we may choose an element from each set. In ZFC, we are only allowed to choose one element from each set S_i in a given family $(S_i)_{i\in I}$.
- All other axioms in ZFC about sets are kept as is.

NBG is very nice because it is a conservative extension of ZFC, meaning every statement about sets provable in NBG could be proved in ZFC, and everything beyond that strictly requires NBG. Moreover, with some clever reductions and seven additional class creation axioms, one can prove the Class comprehension schema so that NBG is finitely axiomatisable. Anyway, the answer to our problem is:

The "set of all sets" is, in fact, a proper class!

To gain some intuition for classes, we remark a few facts. Firstly, all constructions only involving sets again result in sets, e. g., subsets, powersets, unions, products of sets. Recall that a class function $f\colon C \to D$ is just a class $\mathrm{Graph}(f)$ of pairs[1] $(c, f(c)) \in C \times D$ with exactly one such pair for each $c \in C$. The *Axiom of replacement* states that if $f\colon C \to D$ is a function on a set C, then its image $f(C)$ is again a set. In particular, if $f\colon C \xrightarrow{\sim} D$ is a bijection with a set D, then f^{-1} shows that C is a set too, and if $f\colon C \hookrightarrow D$ is an inclusion into a set D, then C is a set as well. On the other hand:

If $f\colon C \to D$ is a function on a proper class C, then $f = \mathrm{Graph}(f)$ is a proper class.

Indeed, define the surjective function $\mathrm{Graph}(f) \to C$ by projecting to the first entry of $(c, f(c))$ (which exists by the Class comprehension schema). If $\mathrm{Graph}(f)$ were a set, then C would be a set, a contradiction.

Solution 2: The "set of all sets" does not consider *all* sets

The second approach was first brought up by Grothendieck in 1958 during Bourbaki discussions and was expounded in *Séminaire de géométrie algébrique 4* (SGA 4, 1972). The idea is to

[1] For the pedantic, the *Kuratowski definition* of pairs defines $(a, b) := \{a, \{a, b\}\}$ to make everything class-based.

simply fix a universe in ZFC, instead of postulating one in NBG. As far as I can tell, this is the standard approach when studying topos theory or modern algebraic geometry based on SGA. My definition here follows the nLab article [5]. For a very readable and more thorough account closer to SGA 4, see [4].

Definition (Grothendieck universe)

A set $\mathcal{U}$ is a *Grothendieck universe* if it has the following properties:

1. Transitivity: if $u \in \mathcal{U}$ and $v \in u$, then $v \in \mathcal{U}$.
2. Power set: if $u \in \mathcal{U}$, then $\mathcal{P}(u) \coloneqq 2^u \in \mathcal{U}$.
3. Unions: if $I \in \mathcal{U}$ and $u_i \in \mathcal{U}$ for each $i \in I$, then $\bigcup_{i \in I} u_i \in \mathcal{U}$.
4. Infinity: $\mathbb{N} \in \mathcal{U}$ where the natural numbers are $0 \coloneqq \varnothing$ and $n \coloneqq \{\varnothing, n-1\}$.

Elements of $\mathcal{U}$ are called *$\mathcal{U}$-small sets*, subsets of $\mathcal{U}$ are called *$\mathcal{U}$-moderate sets*.

Fixing a Grothendieck universe $\mathcal{U}$, one can then show that every standard set-theoretic construction like subsets, families of sets, disjoint unions, intersections, cartesian products and functions are possible inside $\mathcal{U}$, which is sufficient for most applications in algebraic geometry. Indeed, observe that a fixed Grothendieck universe $\mathcal{U}$ gives a model for the NBG axioms: just define sets to be $\mathcal{U}$-small sets and classes to be $\mathcal{U}$-moderate sets.

The actual power of Grothendieck universes is due to the following additional axiom to ZFC.

Axiom (Grothendieck universe axiom)

For each set, there is a Grothendieck universe containing it.

Thus, whenever a universe $\mathcal{U}$ becomes too small, we may always enlarge $\mathcal{U}$ by a universe $\mathcal{V} \ni \mathcal{U}$, whence $\mathcal{V} \supseteq \mathcal{U}$. If desired, one can even stratify to an infinite tower of ever larger universes.[2] This is not possible in NBG since proper classes cannot be contained in classes. It will be apparent that this axiom is very (maybe too) powerful. Anyway, the solution to our problem in a fixed Grothendieck universe $\mathcal{U}$ is:

The "set of all sets" should be understood as the set of all $\mathcal{U}$-small sets, which is $\mathcal{U}$!

Refresher: basic notions of category theory

We recall some basic notions. As is common in most of category theory, we will use the language of classes and NBG.

[2]In even more modern areas using the language of ∞-*categories* by Lurie, size issues are solved by virtue of cardinality. Instead of Grothendieck universes, one uses *uncountable strongly inaccessible cardinalities* κ and defines a set S to be κ-*small* if $\#S < \kappa$ (the properties on κ are only necessary so that $\#S < \kappa$ if and only if $\#\mathcal{P}(S) < \kappa$). For any set S, one can always find such a cardinality $\kappa \geq \#\mathcal{P}(S) > S$ for which S is κ-small, and the mechanics are similar to Grothendieck universes. See [3, Tag 03PP] for more details.

Definition (category)

A *category* $\mathcal{C}$ consists of:

- a class $\mathrm{Ob}(\mathcal{C})$ of *objects*;
- for two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ a class $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of *morphisms*, often written as $f\colon X \to Y$;
- a *composition* operator $\circ$ on morphisms such that if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms, then $g \circ f\colon X \to Z$ is a morphism in $\mathcal{C}$;

such that:

- composition is associative, i. e., $(h \circ g) \circ f = h \circ (g \circ f)$ for all composable f, g, h;
- for each $X \in \mathrm{Ob}(\mathcal{C})$, there is a (unique) *identity* morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ such that $f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f = f$ for all morphisms $f\colon X \to Y$.

We denote by $\mathrm{Mor}(\mathcal{C})$ the class of all morphisms in $\mathcal{C}$.

We usually write $X \in \mathcal{C}$ instead of $X \in \mathrm{Ob}(\mathcal{C})$. If the language of Grothendieck universes is preferred, as we will sometimes do, the above definition can be reformulated by replacing all instances of "class" by "set". In the presence of a universe $\mathcal{U}$, we would rather speak of "$\mathcal{U}$-moderate sets" (subsets of $\mathcal{U}$) instead of arbitrary "sets".

A morphism $f\colon X \to Y$ is an *isomorphism* if there exists an inverse $f^{-1}\colon Y \to X$ such that $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$.

Definition (functor)

A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ between two categories consists of:

- a map $F\colon\ \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$, $X \mapsto F(X)$ on objects;
- a map $F\colon\ \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$, $f \mapsto F(f)$ on morphisms;

such that:

- $F(g \circ f) = F(g) \circ F(f)$ for all composable morphisms f and g in $\mathcal{C}$;
- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all identity morphisms id_X in $\mathcal{C}$.

We avoid any notions of covariance and contravariance. Instead, a contravariant functor F that swaps the direction of $f\colon X \to Y$ in $\mathcal{C}$ to $F(f)\colon F(Y) \to F(X)$ in $\mathcal{D}$ is a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ where $\mathcal{C}^{\mathrm{op}}$ is the *opposite category* obtained from $\mathcal{C}$ by reversing the direction of all morphisms.

Definition (natural transformation)

A *natural transformation* $\eta\colon F \to G$ between two functors $\mathcal{C} \to \mathcal{D}$ consists of a family of morphisms $(\eta_X \colon F(X) \to G(X))_{X \in \mathcal{C}}$ such that the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd} \tag{1}$$

commutes for all morphisms $f \colon X \to Y$ in $\mathcal{C}$, meaning that $\eta_Y \circ F(f) = G(f) \circ \eta_X$.

A natural transformation $\eta \colon F \to G$ such that η_X is an isomorphism for all $X \in \mathcal{C}$ is called a *natural isomorphism*. We will also write $F \cong G$.

Cheating the Yoneda lemma

Yoneda's size issues

To build the celebrated lemma in category theory, we need to exhibit the *hom-functor*. Let $\mathcal{C}$ be a category and let $X \in \mathcal{C}$ be an object. Any morphism $f \colon A \to B$ induces an obvious map $- \circ f \colon \operatorname{Hom}_{\mathcal{C}}(B, X) \to \operatorname{Hom}_{\mathcal{C}}(A, X)$ by pre-composition, and this map is functorial because

$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{C}}(C, X) \arrow[r, "-\circ g"] \arrow[rr, bend right, "-\circ(g \circ f)"'] & \operatorname{Hom}_{\mathcal{C}}(B, X) \arrow[r, "-\circ f"] & \operatorname{Hom}_{\mathcal{C}}(A, X) \end{tikzcd}$$

commutes for all morphisms $f \colon A \to B$ and $g \colon B \to C$.

The next logical step is to define a functor $\operatorname{Hom}_{\mathcal{C}}(-, X)$. Since it reverses the direction of morphisms, it is a functor from $\mathcal{C}^{\mathrm{op}}$ to ... which category? The problem is that $\operatorname{Hom}_{\mathcal{C}}(A, X)$ is generally a class, but there is no category $\mathcal{S}$ of possibly proper classes since the class $\operatorname{Ob}(\mathcal{S})$ cannot contain a proper class. But there is certainly the *category* Set *of all sets*. Let us introduce some terminology.

Definition (sizes of catgories)

Let $\mathcal{C}$ be a category.

- If $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ is a set for all $X, Y \in \mathcal{C}$, then $\mathcal{C}$ is a *locally small* category.
- If $\mathcal{C}$ is locally small and if $\operatorname{Ob}(\mathcal{C})$ is a set, then $\mathcal{C}$ is a *small* category.

For example, Set is locally small because $\operatorname{Hom}_{\mathsf{Set}}(X, Y) \subseteq \mathcal{P}(X \times Y)$ for all sets X and Y. Note that $\mathcal{C}$ is small if and only if $\operatorname{Mor}(\mathcal{C}) = \bigsqcup_{X,Y \in \mathcal{C}} \operatorname{Hom}_{\mathcal{C}}(X, Y)$ is a set since there is an injective map $\operatorname{Ob}(\mathcal{C}) \hookrightarrow \operatorname{Mor}(\mathcal{C})$, $X \mapsto \mathrm{id}_X$. We will say that $\mathcal{C}$ is *large* to suggest that it might not be locally small. For some mathematicians, a category is *by definition* always locally small, as most categories in practice are. As we will see, however, this still leaves size issues.

Hence, in order for $\mathrm{Hom}_{\mathcal{C}}(-, X)$ to be a functor $\mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$, the category $\mathcal{C}$ must be locally small! Since $\mathrm{Hom}_{\mathcal{C}}(-, X)$ is also functorial in the entry X by means of post-composition, we might be tempted to define a functor $X \mapsto \mathrm{Hom}_{\mathcal{C}}(-, X)$ from $\mathcal{C}$ to ... which category again? We somehow need a category of functors $F \colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$. But observe that F is essentially a map

$$\mathrm{Ob}(\mathcal{C}) \cup \bigcup_{X,Y \in \mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Ob}(\mathsf{Set}) \cup \mathrm{Mor}(\mathsf{Set}).$$

If $\mathcal{C}$ is not small, then F is a proper class and cannot be an object of any category! Thus, the category $\mathcal{C}$ must be even small!

Definition (functor category)

Let $\mathcal{C}$ be a small category and $\mathcal{D}$ be a category. The *functor category* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ has functors $\mathcal{C} \to \mathcal{D}$ as objects and natural transformations of such functors as morphisms. Composition of natural transformations $\eta \colon F \to G$ and $\varepsilon \colon G \to H$ is defined by $(\varepsilon \circ \eta)_X = \varepsilon_X \circ \eta_X$ for all $X \in \mathcal{C}$.

Thus, for $\mathcal{C}$ small we obtain the *Yoneda embedding*

$$\mathrm{h}_{\mathcal{C}} \colon \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}), \qquad (X \xrightarrow{f} Y) \longmapsto (\mathrm{Hom}_{\mathcal{C}}(-, X) \xrightarrow{f \circ -} \mathrm{Hom}_{\mathcal{C}}(-, Y)).$$

Lemma 1 (Yoneda lemma)

Let $\mathcal{C}$ be a small category. For each functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ and each object $X \in \mathcal{C}$, there is a bijection of sets

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})}(\mathrm{h}_{\mathcal{C}}(X), F) \xrightarrow{\sim} F(X), \qquad \eta \longmapsto \eta_X(\mathrm{id}_X)$$

which is natural in X and F, i. e., this defines a natural isomorphism of functors $\mathcal{C}^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}) \to \mathsf{Set}$.

Proof idea. This is not surprising. By naturality of η, for all $f \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ we have

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X, X) \arrow[r, "- \circ f"] \arrow[d, "\eta_X"'] & \mathrm{Hom}_{\mathcal{C}}(Y, X) \arrow[d, "\eta_Y"] \\ F(Y) \arrow[r, "F(f)"] & F(X), \end{tikzcd} \qquad \begin{tikzcd} \mathrm{id}_X \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] \\ \eta_X(\mathrm{id}_X) \arrow[r, mapsto] & \eta_Y(f). \end{tikzcd}$$

So, we must define $\eta_Y(f) \coloneqq F(f)(\eta_X(\mathrm{id}_X))$ for each $Y \in \mathcal{C}$, and η is completely determined by the value $\eta_X(\mathrm{id}_X)$. For a full proof, see [1, Proposition 1.4.3, 7, Theorem 2.2.4]. □

Corollary 2 (Yoneda principle)

If $\mathrm{Hom}_{\mathcal{C}}(-,X) \cong \mathrm{Hom}_{\mathcal{C}}(-,Y)$ are naturally isomorphic functors, then there is a unique isomorphism $X \cong Y$ in $\mathcal{C}$ inducing the given natural isomorphism.

Proof. We have a bijection

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathsf{Set})}(\mathrm{h}_{\mathcal{C}}(X), \mathrm{h}_{\mathcal{C}}(Y)) \cong \mathrm{h}_{\mathcal{C}}(Y)(X) = \mathrm{Hom}_{\mathcal{C}}(X,Y).$$

A basic fact from category theory[3] shows that if $\mathrm{h}_{\mathcal{C}}(X) \cong \mathrm{h}_{\mathcal{C}}(Y)$ on the left-hand side, then $X \cong Y$ on the right-hand side. The bijection also shows that the isomorphism $f\colon X \to Y$ is the unique one inducing the isomorphism $\mathrm{h}_{\mathcal{C}}(f)\colon \mathrm{h}_{\mathcal{C}}(X) \to \mathrm{h}_{\mathcal{C}}(Y)$. □

> Whenever you want to appeal to the Yoneda lemma, your category must be small.[4]

What would happen if we assume Grothendieck universes? Suppose that our category $\mathcal{C}$ is too large with respect to a universe $\mathcal{U}$, meaning $\mathrm{Mor}(\mathcal{C}) \subseteq \mathcal{U}$ is $\mathcal{U}$-moderate but not $\mathcal{U}$-small. Then for any universe $\mathcal{V} \ni \mathcal{U}$, the set $\mathrm{Mor}(\mathcal{C}) \in \mathcal{V}$ is $\mathcal{V}$-small and hence $\mathcal{C}$ is small with respect to $\mathcal{V}$. Moreover, the category $\mathsf{Set}_{\mathcal{U}}$ of all $\mathcal{U}$-small sets is a full subcategory of $\mathsf{Set}_{\mathcal{V}}$, meaning any map of $\mathcal{U}$-small sets also exists in $\mathsf{Set}_{\mathcal{V}}$. Hence, we have an embedding $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}_{\mathcal{U}}) \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}_{\mathcal{V}})$. Thus, we may apply the Yoneda lemma to functors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}_{\mathcal{U}} \hookrightarrow \mathsf{Set}_{\mathcal{V}}$ using the Yoneda embedding $\mathrm{h}_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}_{\mathcal{U}}) \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}_{\mathcal{V}})$.

> Assuming the Grothendieck universe axiom, we can *always* appeal to the Yoneda lemma, even if the category is $\mathcal{U}$-large, at the expense that the natural bijection is one of $\mathcal{V}$-small sets.

The Yoneda lemma in form of corollary 2 gives rise to many elegant proofs since instead of studying an object X, we may study the hom-functor $\mathrm{Hom}_{\mathcal{C}}(-,X)$. This perspective shifts from understanding X to understanding how everything relates to X.

Definition (representable functors)

A functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ with $\mathcal{C}$ locally small is *represented* by $X \in \mathcal{C}$ if there is a natural isomorphism $F \cong \mathrm{Hom}_{\mathcal{C}}(-,X)$.

[3] Fully faithful functors are *conservative* or *reflect isomorphisms*, i. e., if $\mathrm{h}_{\mathcal{C}}(f)$ is an isomorphism, then so is f.

[4] You might think that we could generalise the Yoneda lemma to a bijection of classes instead of sets in order to allow for locally small $\mathcal{C}$ like in [7, Theorem 2.2.4]. However, by the same token, a natural transformation of functors $\mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ is a function $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Mor}(\mathsf{Set})$, which is a proper class if $\mathrm{Ob}(\mathcal{C})$ is. By stating naturality as commutativity of naturality squares, the Yoneda lemma now holds for $\mathcal{C}$ with $\mathrm{Ob}(\mathcal{C})$ a set, but not for locally small $\mathcal{C}$. In order to generalise to locally small $\mathcal{C}$, one must avoid any classes of functors in its formulation.

By corollary 2, assuming that $\mathcal{C}$ is small, the representative X of F is unique up to unique isomorphism.

Yoneda-style definitions

There are many examples where a categorical notion has two definitions: one using a Yoneda-style natural isomorphism of hom-functors, and one more abstract definition. You have probably seen the following example already.

Example (adjunction). Let $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ be two functors. Then L is *left adjoint* to R and R is *right adjoint* to L, written $L \dashv R$, if there is a bijection of classes

$$\mathrm{Hom}_{\mathcal{D}}(L(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, R(Y)) \tag{2}$$

for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ such that it is natural in X and Y. Naturality means that the naturality squares in form of eq. (1) induced by any morphisms in $\mathcal{C}$ and $\mathcal{D}$ commutes. Although this bijection of classes works in full generality, it is easiest to work with if this is a natural isomorphism of functors. To this end, we require $\mathcal{C}$ and $\mathcal{D}$ to be *locally small*, in which case eq. (2) is an isomorphism of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathsf{Set}$.
The more general definition requires two natural transformations $\eta\colon \mathrm{id}_{\mathcal{C}} \to R \circ L$ and $\varepsilon\colon L \circ R \to \mathrm{id}_{\mathcal{D}}$ called *unit* and *counit*, respectively. Then $L \dashv R$ are adjoint if the unit and counit satisfy the *triangle identities*

$$\begin{tikzcd} R \arrow[r, "\eta R"] \arrow[dr, "\mathrm{id}_R"'] & R \circ L \circ R \arrow[d, "R\varepsilon"] \\ & R \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} L \arrow[r, "L\eta"] \arrow[dr, "\mathrm{id}_L"'] & L \circ R \circ L \arrow[d, "\varepsilon L"] \\ & L. \end{tikzcd}$$

We obtain $\eta_X\colon X \to R(L(X))$ as the element on the right-hand side of eq. (2) corresponding to $\mathrm{id}_{L(X)}$, and dually obtain $\varepsilon_Y\colon L(R(Y)) \to Y$ as the element on the left-hand side of eq. (2) corresponding to $\mathrm{id}_{R(Y)}$.
An important fact is that right adjoints are unique up to unique isomorphism. This either follows from corollary 2 with $\mathcal{C}$ small, or by a very tedious calculation with (co)units, see [7, Proposition 4.3.1].

I want to present a lesser known definition of limits in Yoneda-style. A functor $F\colon I \to \mathcal{C}$ is called a *diagram* in $\mathcal{C}$ indexed by I. We first review the usual definition via cones.

Definition (limits via cones)

Let $F\colon I \to \mathcal{C}$ be a diagram. A *cone* over F with tip $X \in \mathcal{C}$ is a tuple $(X, f_i\colon X \to F(i))_{i\in I}$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\ F(i) \arrow[rr, "F(i\to j)"] & & F(j) \end{tikzcd}$$

commutes for all $i \to j$ in I. The *limit cone* of F is a cone $(\lim_{i\in I} F(i), \lambda_i)_{i\in I}$ satisfying the following universal property: any cone $(X, f_i)_{i\in I}$ factorises uniquely through the limit cone, i. e., there exists a unique morphism $X \to \lim_{i\in I} F(i)$ such that the diagram

$$\begin{tikzcd} & X \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right, "f_i"'] \arrow[ddr, bend left, "f_j"] & \\ & \lim_{i\in I} F(i) \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\ F(i) \arrow[rr, "F(i\to j)"] & & F(j) \end{tikzcd}$$

commutes for all $i \to j$ in I. We call $\lim_{i\in I} F(i)$ the *limit* of F.

To get to the Yoneda-style definition, we first need to determine when $\mathrm{Fun}(I^{\mathrm{op}}, \mathcal{C})$ is locally small (the I^{op} instead of I is just a notational convention due to the Yoneda lemma) when I is small. A natural transformation $F \to G$ of functors $I^{\mathrm{op}} \to \mathcal{C}$ is a family of morphisms $(F(i) \to G(i))_{i\in I}$, whence

$$\mathrm{Hom}_{\mathrm{Fun}(I^{\mathrm{op}},\mathcal{C})}(F, G) \subseteq \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(F(i), G(i)).$$

Thus, $\mathcal{C}$ must be locally small for the right-hand side to be a set. We say that a diagram $F\colon I^{\mathrm{op}} \to \mathcal{C}$ is *small* if I is small.

Definition (limits as representatives)

Let $F\colon I^{\mathrm{op}} \to \mathsf{Set}$ be a small set-valued diagram. Then its *limit* is

$$\lim_{i\in I} F(i) \coloneqq \mathrm{Hom}_{\mathrm{Fun}(I^{\mathrm{op}},\mathsf{Set})}(\mathrm{pt}, F) \in \mathsf{Set}.$$

Here, $\mathrm{pt}\colon I^{\mathrm{op}} \to \mathsf{Set}$ is the constant functor mapping everything to the singleton set $\{\mathrm{pt}\} \in \mathsf{Set}$. Now, let $F\colon I^{\mathrm{op}} \to \mathcal{C}$ be a small diagram valued in a locally small category $\mathcal{C}$. A *limit* of F is a representative $\lim_{i\in I} F(i) \in \mathcal{C}$ of the functor $\lim_{i\in I} \mathrm{Hom}_{\mathcal{C}}(-, F(i))$, i. e., there is a natural isomorphism

$$\lim_{i\in I} \operatorname{Hom}_{\mathcal{C}}(-, F(i)) \cong \operatorname{Hom}_{\mathcal{C}}\big(-, \lim_{i\in I} F(i)\big).$$

How are these definitions the same? First, let $F\colon I^{\mathrm{op}} \to \mathsf{Set}$ be a set-valued diagram. Then each natural transformation $\eta\colon \mathrm{pt} \to F$ corresponds to a family of elements $(\eta_i(\mathrm{pt}))_{i\in I} \in \prod_{i\in I} F(i)$ that are compatible under the image of F. So, $\lim_{i\in I} F(i)$ is the tip of a limit cone which can be checked by chasing an element $\eta \in \lim_{i\in I} F(i)$ through the cone diagram. For general diagrams $F\colon I^{\mathrm{op}} \to \mathcal{C}$, each $\eta \in \lim_{i\in I} \operatorname{Hom}_{\mathcal{C}}(X, F(i))$ similarly corresponds to a family of morphisms $(\eta_i(\mathrm{pt})\colon X \to F(i))_{i\in I}$, constituting a cone $(X, \eta_i(\mathrm{pt}))_{i\in I}$. The representability of the functor $\lim_{i\in I} \operatorname{Hom}_{\mathcal{C}}(-, F(i))$ encodes the universal property of $\lim_{i\in I} F(i)$.

Example (products). The simplest example of limits is the *product* of a family $(X_i)_{i\in I}$ of objects $X_i \in \mathcal{C}$ indexed by a class I. Consider I as a discrete category, i. e., adding id_i for each $i \in I$ and no other morphisms. If $F\colon I \to \mathcal{C}$, $i \mapsto X_i$, then we write $\prod_{i\in I} X_i \coloneqq \lim_{i\in I} F(i)$, if it exists.

Two kinds of proofs

Let us present three properties of limits and how to prove them with either definition.

Lemma 3 (limits are essentially unique)

Limits of a diagram $F\colon I \to \mathcal{C}$ are, if they exist, unique up to unique isomorphism.

Proof using cones. Let Λ and Λ' be two limits of F. Their universal properties give two morphisms $f\colon \Lambda \to \Lambda'$ and $g\colon \Lambda' \to \Lambda$. If we consider the limit cone of Λ twice, we obtain a diagram

$$\begin{tikzcd}
& \Lambda \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
& \Lambda \arrow[dl] \arrow[dr] & \\
F(i) \arrow[rr, "F(i\to j)"] & & F(j).
\end{tikzcd}$$

Both $g\circ f\colon \Lambda \to \Lambda' \to \Lambda$ and $\mathrm{id}_\Lambda\colon \Lambda \to \Lambda$ fit into this diagram as the dashed arrow, so they must be the same. We similarly show that $f \circ g = \mathrm{id}_{\Lambda'}$, whence $f\colon \Lambda \xrightarrow{\sim} \Lambda'$ is an isomorphism. $\square$

Proof using Yoneda. This follows from the uniqueness of representatives (corollary 2). $\square$

Proposition 4 (hom-functors preserve limits)

Let $\mathcal{C}$ be locally small. If a diagram $F\colon I \to \mathcal{C}$ has a limit in $\mathcal{C}$, then we have a canonical bijection of sets

$$\operatorname{Hom}_{\mathcal{C}}\bigl(X, \lim_{i\in I} F(i)\bigr) \cong \lim_{i\in I} \operatorname{Hom}_{\mathcal{C}}(X, F(i))$$

for all $X \in \mathcal{C}$ which is natural in X.

Local smallness is only needed in order to take the limit on the right-hand side in a category, namely Set.

Proof using cones. By lemma 3, we need to check that $\operatorname{Hom}_{\mathcal{C}}(X, \lim_{i\in I} F(i))$ satisfies the universal property of the diagram $\operatorname{Hom}_{\mathcal{C}}(X, F)$. Let $(S, f_i)_{i\in I}$ be a cone over $\operatorname{Hom}_{\mathcal{C}}(X, F)$. Then pointwise for each $s \in S$, we obtain a cone $(X, f_i(s))_{i\in I}$ over F. By the universal property of $\lim_{i\in I} F(i) \in \mathcal{C}$, there is a unique morphism $\phi_s\colon X \to \lim_{i\in I} F(i)$ through which $f_i(s)$ factorises for each $i \in I$. Hence, these assemble to a morphism $\phi\colon S \to \operatorname{Hom}_{\mathcal{C}}(X, \lim_{i\in I} F(i))$, $s \mapsto \phi_s$ making everything commute. One can now verify uniqueness of ϕ and check that the isomorphism is natural in X by a diagram chase. $\square$

Proof using Yoneda. By definition. $\square$

Proposition 5 (right adjoints preserve limits, RAPL)

Let $L \dashv R$ be adjoint functors $\mathcal{C} \rightleftarrows \mathcal{D}$. If a diagram $F\colon I \to \mathcal{D}$ has a limit in $\mathcal{D}$, then there is a canonical isomorphism

$$R\bigl(\lim_{i\in I} F(i)\bigr) \cong \lim_{i\in I} R(F(i)).$$

Proof using cones. Let $\Lambda := \lim_{i\in I} F(i)$ and let $(\Lambda, \lambda_i\colon \Lambda \to F(i))_{i\in I}$ be the limit cone over F. By lemma 3, we need to show that $(R(\Lambda), R(\lambda_i))_{i\in I}$ is a limit cone over $R \circ F\colon I \to \mathcal{C}$. Let $(X, f_i)_{i\in I}$ be a cone over $R \circ F$ in $\mathcal{C}$. By the adjunction isomorphism, $(L(X), L(f_i))_{i\in I}$ is a cone over F in $\mathcal{D}$, so it factorises through a unique morphism $L(X) \to \Lambda$ by the universal property of Λ. Again by adjunction, we obtain a unique morphism $X \to R(\Lambda)$ through which our original cone $(X, f_i)_{i\in I}$ factorises. $\square$

Proof using Yoneda. A one-liner. By proposition 4, we have natural isomorphisms

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}}\bigl(-, R\bigl(\lim_{i\in I} F(i)\bigr)\bigr) &\cong \operatorname{Hom}_{\mathcal{D}}\bigl(L(-), \lim_{i\in I} F(i)\bigr) \cong \lim_{i\in I} \operatorname{Hom}_{\mathcal{D}}(L(-), F(i)) \\ &\cong \lim_{i\in I} \operatorname{Hom}_{\mathcal{C}}(-, R(F(i))) \cong \operatorname{Hom}_{\mathcal{C}}\bigl(-, \lim_{i\in I} R(F(i))\bigr).\end{aligned}$$

The statement follows from the uniqueness of representatives (corollary 2). $\square$

> All Yoneda-style proofs (except for proposition 4) either require small diagrams F in small categories $\mathcal{C}$ (and $\mathcal{D}$), or assume the Grothendieck universe axiom. The elegance of these kind of proofs veil the true generality of the statements.

Case study: Existence of Kan extensions

Kan extensions play an important role in category theory: they capture how a functor $F\colon I \to \mathcal{C}$ can be extended along $\phi\colon I \to J$ to a functor $J \to \mathcal{C}$. MacLane, one of the founding fathers of category theory, proclaimed in his book *Categories for the Working Mathematician* (1971):

> All concepts are Kan extensions.

There are many (almost equivalent) definitions for Kan extensions, but we will content ourselves with an adjunction-style definition which is easier to understand. Suffice it to say, there is a more general definition via universal properties without size assumptions on I and J.[5] As in the case of hom-functors, a functor $\phi\colon I \to J$ induces a pullback $\phi^*(G) \coloneqq G \circ \phi\colon I \to \mathcal{C}$ for all functors $G\colon J \to \mathcal{C}$.

> **Definition** (right Kan extensions)
>
> Let $\phi\colon I \to J$ be functors of small categories and let $F\colon I \to \mathcal{C}$ be a functor. The *right Kan extension* of F along ϕ is a functor $\operatorname{Ran}_\phi F\colon J \to \mathcal{C}$ such that there is a bijection of classes
> $$\mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(\phi^*(G), F) \cong \mathrm{Hom}_{\mathrm{Fun}(J,\mathcal{C})}(G, \operatorname{Ran}_\phi F) \tag{3}$$
> for all functors $G\colon J \to \mathcal{C}$ which is natural in G.

In particular, right Kan extensions solve the extension problem

$$\begin{tikzcd}
I \arrow[r, "F"] \arrow[d, "\phi"'] & \mathcal{C} \\
J \arrow[ur, dashed, "\operatorname{Ran}_\phi F"'] \arrow[ur, phantom, shift left=2, "\Uparrow"] &
\end{tikzcd}$$

which commutes up to a natural transformation $\phi^*(\operatorname{Ran}_\phi F) \to F$. Using corollary 2 (if $\mathcal{C}$ is small) or a more general diagram chase, one can show that $\operatorname{Ran}_\phi F$ is unique up to unique natural isomorphism. If $\operatorname{Ran}_\phi F$ exists for all F, this bijection becomes an adjunction $\phi^* \dashv \operatorname{Ran}_\phi$. This leads to the following.

[5] Let $\phi\colon I \to J$ and $F\colon I \to \mathcal{C}$ be functors. A *right Kan extension* of F along ϕ is a functor $\operatorname{Ran}_\phi F\colon J \to \mathcal{C}$ with a natural transformation $\varepsilon_F\colon \phi^*(\operatorname{Ran}_\phi F) \to F$ such that any natural transformation $\alpha\colon \phi^*(G) \to F$ with $G\colon J \to \mathcal{C}$ factorises uniquely through ε_F. This means that there is a unique natural transformation $\beta\colon G \to \operatorname{Ran}_\phi F$ such that $\alpha = \varepsilon_F \circ \phi^*(\beta)$ where $\phi^*(\beta) = \beta\phi$ is whiskering of β and ϕ.

Lemma 6 (limits are Kan extensions)

Let $\phi\colon I \to \{\mathrm{pt}\}$ be the functor into the category $\{\mathrm{pt}\}$ with one object and no non-identity morphism. Then limits are right Kan extensions along ϕ.

Proof idea. Suppose that I is small and that every diagram/functor in $\mathrm{Fun}(I, \mathcal{C})$ has a limit in $\mathcal{C}$ and admits a right Kan extension along ϕ. Then $\lim_{i\in I}\colon \mathrm{Fun}(I, \mathcal{C}) \to \mathcal{C}$ is a functor. Observe that ϕ^* is the functor

$$\phi^*\colon \mathcal{C} \cong \mathrm{Fun}(\{\mathrm{pt}\}, \mathcal{C}) \longrightarrow \mathrm{Fun}(I, \mathcal{C}), \qquad X \longmapsto (I \xrightarrow{\Delta_X} \mathcal{C})$$

where Δ_X is the constant functor $i \mapsto X$, and that a natural transformation $\eta\colon \Delta_X \to F$ is a cone $(X, \eta_i\colon X \to F(i))_{i\in I}$. Hence, we obtain an adjunction $\phi^* \dashv \lim_{i\in I}$:

$$\mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(\Delta_X, F) \cong \mathrm{Hom}_{\mathcal{C}}\big(X, \lim_{i\in I} F(i)\big).$$

With the adjunction $\phi^* \dashv \mathrm{Ran}_\phi$, uniqueness of right adjoints implies that $\lim_{i\in I} \cong \mathrm{Ran}_\phi$. A similar argument works if we only assume that a specific functor $F\colon I \to \mathcal{C}$ has a limit and a right Kan extension (use corollary 2, or do a diagram chase to avoid size restrictions on $\mathcal{C}$). See [6, Proposition 2.2] for more details, or [7, Proposition 6.5.1] for a proof via universal properties. □

We define a category $\mathcal{C}$ to be *complete* if it admits all *small* limits. The following innocent statement is used all the time in category theory.

Lemma 7 (existence of Kan extensions)

Let $\phi\colon I \to J$ be a functor from a small category I to a locally small category J, and let $F\colon I \to \mathcal{C}$ be functor into a complete category $\mathcal{C}$. Then the right Kan extension $\mathrm{Ran}_\phi F$ exists.[6]

Proof idea. Consider the category $(j \downarrow \phi)$ with morphisms $j \to \phi(i)$ as objects and commutative triangles

$$\begin{tikzcd} & j \arrow[dl] \arrow[dr] & \\ \phi(i) \arrow[rr, "\phi(i\to i')"] & & \phi(i') \end{tikzcd}$$

[6] In fact, the Kan extension is a *pointwise Kan extension*, i. e., it can be described pointwise by limits. Such Kan extensions are much better behaved.

as morphisms.[7] The statement is a corollary of the following claim, which holds without any size assumptions. Note that the limits are indexed by $(j \downarrow \phi)$.

> *Claim.* If $\lim_{j \to \phi(i)} F(i)$ exists for all $j \in J$, then $\mathrm{Ran}_\phi F$ exists and is given pointwise by $(\mathrm{Ran}_\phi F)(j) \cong \lim_{j \to \phi(i)} F(i)$ for all $j \in J$.

This is consistent with lemma 6 if $J = \{\mathrm{pt}\}$: we have $\lim_{\mathrm{pt} \to \phi(i)} F(i) \cong \lim_{i \in I} F(i) \cong (\mathrm{Ran}_\phi F)(\mathrm{pt})$. The proof is very technical but straightforward: simply define $\mathrm{Ran}_\phi F$ pointwise as in the claim and, assuming J is small, exhibit eq. (3). For more details, see [1, Theorem 2.2.3]. An, in my opinion, more intuitive proof, using the definition of Kan extensions by universal properties, exploits the universal property of $\lim_{j \to \phi(i)} F(i)$, see [7, Theorem 6.2.1].

Back to the main statment. Since J is locally small, $(j \downarrow \phi)$ is locally small. Since $\mathrm{Ob}(I)$ is a set, $\mathrm{Ob}(j \downarrow \phi)$ is a set. Hence, $(j \downarrow \phi)$ is small and diagrams indexed by it have limits in $\mathcal{C}$. Now, apply the claim. □

Example (Yoneda extensions). It is known that the functor category $\mathrm{Fun}(\mathcal{C}, \mathsf{Set})^{\mathrm{op}}$ is complete.[8] The "opposite" Yoneda embedding $\phi = (\mathrm{h}_{\mathcal{C}^{\mathrm{op}}})^{\mathrm{op}} \colon \mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathsf{Set})^{\mathrm{op}}$ allows us to embed any small category $\mathcal{C}$ into its *free completion* $\mathrm{Fun}(\mathcal{C}, \mathsf{Set})^{\mathrm{op}}$. Applying lemma 7, any functor $F \colon \mathcal{C} \to \mathcal{D}$ into a complete category $\mathcal{D}$ "factorises" (up to a natural transformation) uniquely through the completion ϕ and the right Kan extension $\mathrm{Ran}_\phi F$, i. e., the free completion satisfies the expected universal property.

Uncheatable size restrictions

The size conditions in lemma 7 are crucial: it is impossible for this to be true if I were any larger. By way of contradiction, lemma 6 would imply that for any *large* (or even locally small) diagram $F \colon I \to \mathcal{C}$, the large limit $\lim_{i \in I} F(i) \cong \mathrm{Ran}_\phi F$ with $\phi \colon I \to \{\mathrm{pt}\}$ exists in $\mathcal{C}$. But for $\mathcal{C} = \mathsf{Set}$, this would obstruct the following.

Theorem 8 (Set is only complete)

Set admits all small limits, but not all (large) limits.

Proof. For a small diagram $F \colon I \to \mathsf{Set}$, one can verify that

$$\lim_{i \in I} F(i) \cong \left\{ (x_i)_i \in \prod_{i \in I} F(i) \;\middle|\; F(f)(x_i) = x_j \text{ for all } f \in \mathrm{Hom}_I(i, j) \right\}.$$

[7]More precisely, the *comma category* $(j \downarrow \phi)$ has pairs $(i, j \to \phi(i)) \in \mathrm{Ob}(I) \times \mathrm{Mor}(J)$ as objects, and a morphism $(i, j \to \phi(i)) \to (i', j \to \phi(i'))$ is a morphism $i \to i'$ in I together with the commutative triangle.

[8]The fact is that (co)limits in functor categories $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are computed "pointwise". In particular, it admits the same (co)limits that $\mathcal{D}$ admits. We show in theorem 8 that Set is complete, but Set is also cocomplete.

Note that $\prod_{i \in I} F(i)$ is a set because I is small.

For a large (even locally small) limit, consider the proper class $I = \mathrm{Ob}(\mathsf{Set}) \setminus \{\varnothing\}$ of all non-empty sets, and suppose that the product $\Pi := \prod_{S \in I} S$ exists. Then the limit cone gives a map $\pi_{\mathcal{P}(\Pi)} \colon \Pi \to \mathcal{P}(\Pi)$ which cannot be surjective due to cardinality reasons, say, it misses $p \in \mathcal{P}(\Pi)$. By the Axiom of global choice, choose an element $q_S \in S$ for each $S \in I \setminus \{\mathcal{P}(\Pi)\}$. We obtain a cone $(\{\mathrm{pt}\}, f_S)_{S \in I}$ where $f_S(\mathrm{pt}) = q_S$ for $S \neq \mathcal{P}(\Pi)$ and $f_{\mathcal{P}(\Pi)}(\mathrm{pt}) = p$. By the universal property of Π, the map $f_{\mathcal{P}(\Pi)}$ factorises through $\pi_{\mathcal{P}(\Pi)}$, which is absurd. $\square$

Incidentally, for $I = \mathrm{Ob}(\mathsf{Set})$, the product $\prod_{S \in I} S$ exists. Indeed, any cone over $I \to \mathsf{Set}$ must have the empty set $\varnothing$ at the tip, so $\prod_{S \in I} S = \varnothing$. This means that Set admits some large limits.

The counterexample above reveals that the size of I depends on the size of limits that $\mathcal{C}$ admits. You might say: "Just make $\mathcal{C}$ admit all large limits!" But the following statement due to Freyd dwarves this idea.

Theorem 9 ("large-complete" categories are thin)

Let $\mathcal{C}$ be a category which admits products indexed by $\mathrm{Mor}(\mathcal{C})$. Then $\mathcal{C}$ is a *thin* category, i. e., there is at most one morphism $X \to Y$ for any $X, Y \in \mathcal{C}$.

Proof with cones. By way of contradiction, suppose that there are two distinct morphisms $X \rightrightarrows Y$ in $\mathcal{C}$. By the universal property of $\Pi := \prod_{f \in \mathrm{Mor}(\mathcal{C})} Y$, the class $\mathrm{Hom}_{\mathcal{C}}(X, \Pi)$ is in bijection with cones $(X, f_i \colon X \to Y)_{i \in \mathrm{Mor}(\mathcal{C})}$. Since there are (at least) two choices for each f_i, the cardinality of $\mathrm{Hom}_{\mathcal{C}}(X, \Pi)$ is at least $2^{\#\,\mathrm{Mor}(\mathcal{C})} > \#\,\mathrm{Mor}(\mathcal{C})$. This contradicts $\mathrm{Hom}_{\mathcal{C}}(X, \Pi) \subseteq \mathrm{Mor}(\mathcal{C})$. $\square$

Proof with Yoneda. Let $\mathcal{C}$ be small. Again, suppose that there are $X, Y \in \mathcal{C}$ such that the cardinality of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is at least 2. By proposition 4, the class

$$\mathrm{Hom}_{\mathcal{C}}\Big(X, \prod_{f \in \mathrm{Mor}(\mathcal{C})} Y\Big) \cong \prod_{f \in \mathrm{Mor}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

has cardinality at least $2^{\#\,\mathrm{Mor}(\mathcal{C})} > \#\,\mathrm{Mor}(\mathcal{C})$, contradicting that it is a subclass of $\mathrm{Mor}(\mathcal{C})$. $\square$

For example, since Set is not thin, it cannot admit all products of size $\mathrm{Mor}(\mathsf{Set})$.

To summarise, the size of I in lemma 7 is limited by the size of limits that $\mathcal{C}$ admits. However, almost all (reasonable) categories $\mathcal{C}$ can never admit all large limits, as the size is capped by the size of $\mathrm{Mor}(\mathcal{C})$.

One can impose different smallness conditions on lemma 7 that make it slightly sharper, but one can never eliminate smallness. See the MathOverflow answer [2], which also motivated this section.

However, what if we use Grothendieck universes to complete categories? Recall that a set S is $\mathcal{U}$-small if $S \in \mathcal{U}$, and is $\mathcal{U}$-moderate if $S \subseteq \mathcal{U}$, and recall that subsets of $\mathcal{U}$-small sets are $\mathcal{U}$-small. Consider the category $\mathsf{Set}_{\mathcal{U}}$ of all $\mathcal{U}$-small sets, which admits all $\mathcal{U}$-small limits but not all $\mathcal{U}$-moderate limits by theorem 8. If we want to complete $\mathsf{Set}_{\mathcal{U}}$ by all $\mathcal{U}$-moderate limits, we can choose any universe $\mathcal{V} \ni \mathcal{U}$. Any $\mathcal{U}$-moderate set becomes $\mathcal{V}$-small and hence $\mathcal{U}$-moderate limits exist in $\mathsf{Set}_{\mathcal{V}}$ by theorem 8. But again, $\mathsf{Set}_{\mathcal{V}}$ does not admit all $\mathcal{V}$-moderate limits. No matter how large universes $\mathcal{U} \in \mathcal{V} \in \cdots$ we use to complete the category $\mathsf{Set}_{\mathcal{U}}$, absolute completeness escapes us. Taking the union $\mathfrak{U}$ of all the universes $\mathcal{U}, \mathcal{V}, \ldots$ (which is generally not a universe) and picking a universe $\mathfrak{V} \ni \mathfrak{U}$ still does not help: $\mathsf{Set}_{\mathfrak{V}}$ does not admit all $\mathfrak{V}$-moderate limits.

Finally, I would like to come back to the title of this essay. When contesting the existence of God by an infinite regress argument, Russell wrote in *Why I am not Christian* (1927):

> If everything must have a cause, then God must have a cause. If there can be anything without a cause, it may just as well be the world as God, so that there cannot be any validity in that argument. It is exactly of the same nature as the Hindu's view, that the world rested upon an elephant and the elephant rested upon a tortoise; and when they said, "How about the tortoise?" the Indian said, "Suppose we change the subject."

The answer: It's turtles all the way down!

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